10.

Differential equations

- Differential equations
- Differential equations of the form $\frac{dA}{dt} = kA$, or $\dot{x} = kx$
- The logistic model for growth
- Can we get some idea of what the relationship between x and y will look like from the differential equation directly, without integrating?
- Slope fields
- Euler's method
- Miscellaneous exercise ten

Situation One

(You should manage this situation if you recall the work on growth and decay from *Mathematics Methods* Unit Three.)

Anaesthetic is administered to a patient. The patient's body 'uses up' this anaesthetic such that if P mg is in the patient's body at time t then

$$\frac{dP}{dt} = -0.022P \text{ mg/min.}$$

A top-up dose is required when the amount in the patient's body falls to 60% of the initial amount administered.

What time after the initial amount is administered, to the nearest minute, is the top-up dose necessary?

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Situation Two

If we were given that	$\frac{dy}{dx} = 3x^2 + 4x - 1$
we can obtain	$y = x^3 + 2x^2 - x + c.$
Further, if we are told that when	$x = 1, \quad y = 5$
we can determine that	$c = 3.$
Hence	$y = x^3 + 2x^2 - x + 3.$
However, suppose we are given	$\frac{dy}{dx} = 3x^2y, \quad y > 0,$
and that when	$y = 1, \quad x = 1$

can you now find an expression for y in terms of x?

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As suggested on the previous page, you should have managed the first situation from your work on growth and decay in *Mathematics Methods* Unit Three.

With the second situation, did the fact that having differentiated y with respect to x and still having y in the answer make you think that y could be an exponential function (because they differentiate to expressions involving themselves)?

WS
Simple differential equations

Differential equations

Any equation that involves one or more derivatives, e.g. $\frac{dy}{dx}, \frac{dp}{dt}, \frac{dv}{dp}, \frac{d^2y}{dx^2}$ etc, is called a **differential equation**.

The highest **order** of derivative featured gives the **order** of the differential equation.

For example, $\frac{dy}{dx} = 2x + 3$ is a first order differential equation.

To *solve* a differential equation we must find a relationship between the variables involved, that satisfies the differential equation, but that does not contain any derivatives.

To solve $\frac{dy}{dx} = 2x + 3$ we integrate both sides with respect to x.

This gives $y = x^2 + 3x + c$

Note

Integrating both sides will allow us to solve any differential equation that is of the form

$$f(y)\frac{dy}{dx} = g(x)$$

(or that can be expressed in that form).

For example, to solve $y\frac{dy}{dx} = 4x + 1$ we integrate both sides with respect to x:

$$\int y\frac{dy}{dx}\,dx = \int (4x+1)\,dx$$

i.e. $$\int y\,dy = \int (4x+1)\,dx$$

$\therefore$ $$\frac{y^2}{2} = 2x^2 + x + c$$

Note
- The reader should confirm that differentiating this equation with respect to x does give the original differential equation.
- With the '+ c' involved we have found the *family* of solutions to the differential equation. We call this the **general solution**. Given more information that allows 'c' to be determined, we can find a **particular solution**, as the next example will show.

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- Though we have integrated each side of the equation it is *not* necessary for our answer to include two constants, as shown below.

$$\int y\,dy = \int (4x+1)\,dx$$
$$\frac{y^2}{2} + c_1 = 2x^2 + x + c_2$$
$$\frac{y^2}{2} = 2x^2 + x + c \qquad \text{where } c = c_2 - c_1.$$

- In going from $y\dfrac{dy}{dx} = 4x + 1$ to $\int y\,dy = \int (4x+1)\,dx$ we are **separating the variables**.

  The process can be remembered by thinking of $\dfrac{dy}{dx}$ as a fraction and rearranging to 'put the ys with the dy and the xs with the dx', and then integrating.

- Some calculators are able to solve differential equations. Explore the capability of your calculator in this regard.

  (If solving a second (or higher) order differential equation, e.g. $y'' = 6x$ the general solution will involve more than one constant. Hence, when using a calculator to solve a differential equation the calculator might number the constants c1, c2, …)

```
deSolve(y · y′ = 4 · x + 1, x, y)
                 y² = 4 · x² + 2 · x + c1

deSolve(y″ = 6 · x, x, y)
                 y = x³ + c2 · x + c3
```

EXAMPLE 1

Solve $\dfrac{dy}{dx} = \dfrac{3x(x+2)}{2y-1}$, given that when $x = 1, y = 2$.

Solution

We are given: $$\frac{dy}{dx} = \frac{3x(x+2)}{2y-1}$$

Separating the variables and integrating:

$$\int (2y-1)\,dy = \int (3x^2 + 6x)\,dx$$

$\therefore$ $$y^2 - y = x^3 + 3x^2 + c$$

When $x = 1, y = 2$, thus $$2^2 - 2 = 1^3 + 3(1)^2 + c$$
$$2 = 4 + c$$

and so $$c = -2$$

$\therefore$ $$y^2 - y = x^3 + 3x^2 - 2$$

Exercise 10A

Find general solutions for each of the following differential equations.

1 $\frac{dy}{dx} = 8x - 5$

2 $\frac{dy}{dx} = 6\sqrt{x}$

3 $8y\frac{dy}{dx} = 4x - 1$

4 $3y\frac{dy}{dx} = \frac{5}{x^2}$

5 $14x^2y\frac{dy}{dx} = 1$

6 $4x^2 \sin 2y\frac{dy}{dx} = 5$

7 $\frac{dy}{dx} = \frac{(8x+1)}{2y-3}$

8 $\frac{dy}{dx} = \frac{x(2-3x)}{4y-5}$

9 $x^2\frac{dy}{dx} = \frac{1}{\cos y}$

10 $(y^2+1)^5\frac{dy}{dx} = \frac{x}{2y}$

11 Solve $\frac{dy}{dx} = 6x$, given that when $x = -1, y = 4$.

12 Solve $6x^2y\frac{dy}{dx} = 5$, given that when $x = 0.5, y = 1$.

13 Solve $(2 + \cos y)\frac{dy}{dx} = 2x + 3$, given that when $x = 1,\ y = \frac{\pi}{2}$.

14 Solve $\frac{dy}{dx} = \frac{4x(x^2+2)}{2y+3}$, given that when $x = 1, y = 2$.

15 Let us suppose that an object is moving such that its speed, v metres/second ($v \geq 0$), when distance s metres from an origin ($s \geq 0$), is such that

$$v\frac{dv}{ds} = 6s^2.$$

If $v = 6$ when $s = 2$, find v when $s = 3$.

16 A particular curve is such that $\frac{dy}{dx} = -\frac{\sin x}{y}$.

The curve passes through the point $\left(\frac{\pi}{3}, 2\right)$.

a If point A, (π, a), $a > 0$, lies on the curve find a.

b If point B, $\left(\frac{\pi}{6}, b\right)$, $b > 0$, lies on the curve find b and the gradient of the curve at point B.

17 An experiment involves pumping air into a hollow rubber sphere using a machine that responds to the increasing pressure in the sphere by decreasing the amount of air it delivers. Indeed, if the sphere has an initial volume of 20 cm^3, then its volume, V cm^3, t seconds later is such that

$$\frac{dV}{dt} = \frac{25}{2V}.$$

a Find the volume of the sphere when $t = 20$.

b If pumping ceases when $V = 40$ find the value of t when this occurs.

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Differential equations of the form $\frac{dA}{dt} = kA$, or $\dot{x} = kx$

WS Differential equations and exponentials

Situation One at the beginning of this chapter involved an equation of the above form, i.e.

$$\frac{dP}{dt} = -0.022P.$$

If, as the situation suggested, you remembered your work on growth and decay from Unit Three of *Mathematics Methods* you would have recalled that:

$$\text{If } \frac{dP}{dt} = kP \text{ then } P = P_0e^{kt}, \text{ where } P_0 \text{ is the value of } P \text{ when } t = 0.$$

However we can now solve differential equations of this type by **separating the variables** rather than by simply quoting a known solution. The following example demonstrates this separation of variables for such differential equations.

EXAMPLE 2

Figures indicate that a particular country's instantaneous growth rate is always approximately 5% of the population at that time.

a How long does it take the population to double?

b If the population now is 2 000 000 in how many years will it be 8 000 000?

Solution

a If the population after t years is P then we are told that $\frac{dP}{dt} \approx 0.05P$.

Separating the variables $\int \frac{1}{P}\, dP \approx \int 0.05\, dt$

$\therefore$ $\ln P \approx 0.05t + c$ (No need for absolute value as $P > 0$).

Hence $P \approx e^{0.05t + c}$

$= e^{0.05t}e^{c}$

I.e. $P \approx P_0e^{0.05t}$ (where $P_0 = e^c$)

Taking 'now' as $t = 0$ then P_0 is the current population.

If the population is $2P_0$ in T years then $2P_0 \approx P_0e^{0.05T}$

i.e. $2 \approx e^{0.05T}$

Solving by taking natural logarithms, as shown below, or by calculator.

$$\ln 2 \approx 0.05\,T \ln e$$

$$T = \frac{\ln 2}{0.05}$$

$$\approx 14$$

The population will double in approximately 14 years.

b From **a** it follows that if the population now is 2 000 000 it will be 4 000 000 in 14 years and 8 000 000 in a further 14 years.

The population will be 8 000 000 in approximately 28 years.

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Exercise 10B

For questions **1**, **2** and **3** give answers to the nearest whole number.

(The above statement is made to enable you to check your answers, thus confirming correct method. However you might like to consider what more appropriate rounding would be for each of these questions, given the data supplied.)

1 If $\frac{dA}{dt} = 1.5A$, $A > 0$, and $A = 100$ when $t = 0$, find A when

a $t = 1$, **b** $t = 5$.

2 If $\frac{dP}{dt} = 0.25P$, $P > 0$, and $P = 5000$ when $t = 0$, find P when

a $t = 5$, **b** $t = 25$.

3 If $\frac{dQ}{dt} = -0.01Q$, $Q > 0$, and $Q = 100\,000$ when $t = 0$, find Q when

a $t = 20$, **b** $t = 50$.

4 A particular radioactive isotope decays continuously at a rate of 8% per year. Five kilograms of this isotope are produced in a particular industrial process. How much remains undecayed after 25 years?

5 A particular radioactive isotope decays continuously at a rate of 2% per year. Twenty kilograms of this isotope are produced in a particular industrial process. How much remains undecayed after 50 years?

6 Find the half-life of a radioactive element that decays according to the rule:

$\frac{dA}{dt} = -0.0004A$, where A is the amount present after t years.

(The half-life of a radioactive element is the time taken for the amount present to halve.)

7 Cesium-137, a radioactive form of the metal Cesium, decays such that the mass $M(t)$ kg present after t years satisfies the rule $\frac{dM}{dt} = -kM$.

The half-life of a radioactive element is the time taken for the amount present to halve, and for Cesium-137 this is 30 years.

If 1 kg of Cesium-137 is produced in a certain industrial process how much of this remains radioactive after

a 30 years? **b** 60 years? **c** 40 years?

8 The radioactive Uranium isotope U-234 decays such that the mass $M(t)$ kg present after t years satisfies the rule $\frac{dM}{dt} = -kM$.

The half-life of a radioactive element is the time taken for the amount present to halve, and for U-234 this is 250 000 years.

What percentage of an original amount of U-234 remains after 5000 years?

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9 Worldwide numbers of a particular endangered species of animal fell from 325 000 to 56 000 in the space of 8 years. If we take this decline to be such that the instantaneous rate of decline was always approximately p% per annum, find p.

10 All living plants and animals are thought to maintain a constant level of radiocarbon in their bodies. When the plant or animal dies the radiocarbon is no longer absorbed with the intake of air and food and so the radiocarbon levels in the plant or animal declines according to the rule:

$$\frac{dC}{dt} = -kC$$

where $C(t)$ is the mass of radiocarbon t years after death.

The half-life of radiocarbon is 5700 years. If the level of radiocarbon in an animal bone fragment is found to be 60% of the level maintained by the live animal, estimate the number of years ago that the animal died.

11 Following a nuclear leakage an area is designated unsafe for humans due to the level of radioactivity caused by an element, with a half-life 30 years, decaying according to the rule

$$\frac{dM}{dt} = -kM$$

where $M(t)$ is the mass of the radioactive element present at time t years.

The level of radioactivity is found to be 15 times the level considered 'safe'. For how many years after this measurement was taken should the area be considered unsafe?

12 Suppose you wish to determine how long it will take to double the value of an investment if the interest rate is p% per annum, compounded continuously. According to the 'rule of 72' an approximate answer can be obtained by dividing 72 by p.

A more accurate answer would be obtained if a 'rule of 69.3' was used.

a Justify the above statements mathematically.

b If the 'rule of 69.3' gives a more accurate answer than the 'rule of 72' why is it that it is the latter rule that tends to be used?

13 If a hot item is placed in an environment that has a temperature of 28°C and left to cool, the temperature (T°C) of the item t minutes later satisfies the equation

$$\frac{dT}{dt} = -k(T - 28)$$

where k is a positive constant.

Some time after initial placement the temperature of the item was found to be 135°C and, ten minutes later, it was found to be 91°C.

The temperature of the item was known to be 240°C when initially placed in the cooling environment. How long before the 135°C temperature was recorded was the item placed in the 28°C environment?

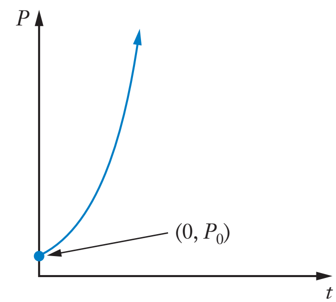
How might the above concept be used by a forensic team to estimate a time of death if they are called to the scene of a recent suspicious death?

The logistic model for growth

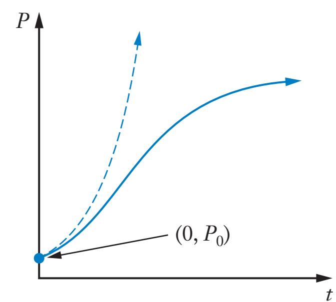
Let us consider further the differential equation

$$\frac{dP}{dt} = kP, \text{ with } k > 0,$$

and its solution $\quad P = P_0 e^{kt}$, shown graphed on the right.



Whilst this model might apply for some growth situations, in many cases the exponential increase might not continue indefinitely in this way. The increase in a population, be it human, animal, bacterial, etc., may well be influenced by other factors such as food supply, available space, predators, etc. For some situations it may be more likely that the initial growth rate will, after a while, decrease and the population may start to head towards some steady, maintainable level. For such situations the unbroken line in the second graph might be a better model to use.



Such a model can be described algebraically by a differential equation of the form:

$$\frac{dP}{dt} = k_1 P - k_2 P^2$$

Or, using y and t for our variables and a and b as the constants:

$$\frac{dy}{dt} = ay - by^2, \text{ with } a > 0 \text{ and } b > 0.$$

This is called **the logistic equation**. It has the general solution

$$y = \frac{a}{b + ce^{-at}} \qquad \text{where } c \text{ is some constant.}$$

Note

The reader should confirm that differentiating this general solution with respect to t does indeed give a differential equation of the required form.

Note that with $\quad y = \dfrac{a}{b + ce^{-at}} \quad$ as $t \to \infty$ then $y \to \dfrac{a}{b}$.

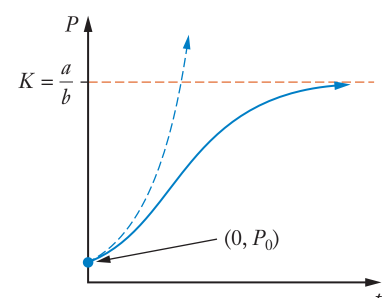
Hence the *levelling-out* or *limiting* population, also called the *carrying capacity*, is $\dfrac{a}{b}$.

For this reason the logistic equation general solution

is sometimes written as $\quad y = \dfrac{K}{1 + Ce^{-at}}$

because, in this form, K is the levelling-out population.

To give the required growth model we require $C > 0$ because then, as $t \to \infty$, the population will approach the limiting value K 'from below'.



The reader should view the graphs of $\quad y = 500e^{0.2x} \quad$ (uninhibited growth)

and $\quad y = \dfrac{5000}{1 + 9e^{-0.2x}} \quad$ (logistic model)

(use $0 \le x \le 30$ and $0 \le y \le 6000$) and see if a 'levelling off' is achieved.

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EXAMPLE 3

Currently (i.e. $t = 0$), N, the number of inmates in a particular prison suffering, or having previously suffered, from a particular form of influenza, is 40.

The value of N is expected to grow such that $\frac{dN}{dt} = \frac{N}{4} - \frac{N^2}{8000}$, with t measured in days.

a The logistic growth model with differential equation $\frac{dy}{dt} = ay - by^2$ has the limiting value of y as $\frac{a}{b}$. Find the limiting value of N according to this model.

b Find $N(t)$, a formula for N as a function of t.

Solution

a Comparing the given expression to $ay - by^2$ we have $a = \frac{1}{4}$ and $b = \frac{1}{8000}$.

Hence the limiting value of N is 2000.

b We are given $$\frac{dN}{dt} = \frac{N}{4} - \frac{N^2}{8000} = \frac{(2000 - N)N}{8000}.$$

Separating the variables $$\int \frac{8000}{(2000 - N)N}\,dN = \int dt$$

Using our understanding of partial fractions we can write this as:

$$\int \left(\frac{1}{N} + \frac{1}{2000 - N}\right) dN = \int \frac{1}{4}\,dt \qquad [1]$$

The limiting value of N is 2000, and N is +ve, so both denominators will be positive.

$\therefore$ $$\ln N - \ln(2000 - N) = 0.25t + c$$

$$\ln\left(\frac{N}{2000 - N}\right) = 0.25t + c$$

$$\frac{N}{2000 - N} = A\,e^{0.25t} \text{ (where } A = e^c\text{)}$$

When $t = 0$, $N = 40$. Hence $A = \frac{1}{49}$.

Substituting for A and re-arranging gives: $$N = \frac{2000}{1 + 49e^{-0.25t}}.$$

Alternatively, had we not noticed that the denominators would be positive:

From [1]: $$\ln|N| - \ln|2000 - N| = 0.25t + c.$$

Hence $$\ln\left|\frac{N}{2000 - N}\right| = 0.25t + c$$

and so $$\left|\frac{N}{2000 - N}\right| = A\,e^{0.25t} \text{ (with } A = e^c\text{)}.$$

Either $\frac{N}{2000 - N} = A\,e^{0.25t}$ or $\frac{N}{2000 - N} = -A\,e^{0.25t}$

In both cases, with $N = 40$ when $t = 0$, we obtain $$\frac{N}{2000 - N} = \frac{1}{49}e^{0.25t}.$$

This rearranges to $N = \frac{2000}{1 + 49e^{-0.25t}}$ as before.

Alternatively we could write:

From [1]: $\ln|N| - \ln|2000 - N| = 0.25t + c.$

Hence $\ln\left|\dfrac{N}{2000 - N}\right| = 0.25t + c$

and so $\dfrac{N}{2000 - N} = \pm A\, e^{0.25t}$ (with $A = e^c$).

$= Ce^{0.25t}$ (with $C = \pm A$).

Using $N = 40$ when $t = 0$ to determine C, substituting for C and rearranging gives:

$$N = \frac{2000}{1 + 49e^{-0.25t}} \quad \text{as before.}$$

The logistic model has applications in a number of areas, as the previous example and some of the questions of the next exercise show.

Exercise 10C

1 When a new technological device is introduced to a country the number of people having the device can grow according to a logistic model. Let us suppose that N, the number of people, in millions, having the device, t months after the monitoring of such numbers began, was such that

$$\frac{dN}{dt} = 0.45N - 0.015N^2.$$

When monitoring commenced 500 000 people had the device (i.e. $N = 0.5$).

Use the fact that a differential equation of the form

$$\frac{dy}{dt} = ay - by^2, \qquad \text{with } a > 0 \text{ and } b > 0,$$

has the general solution $y = \dfrac{a}{b + ce^{-at}}$, where c is some constant, to determine

a the value of c for the take up of device situation,

b the number of people with the device 10 months after the monitoring began.

2 A researcher investigates how quickly a rumour spreads amongst an island community of 150 000 people. When the research commences it is thought that 300 of the 150 000 know the rumour. One day later 920 know it.

Applying a logistic model to this situation, if y is the number of people from the community of 150 000 who know the rumour, t days after research commenced:

$$y = \frac{K}{1 + Ce^{-at}}, \qquad \text{with } K = 150\,000.$$

How many people does the model predict will know the rumour 5 days after the research began (to the nearest 100 people)?

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3 Using separation of variables and partial fractions find the particular growth solution, $y = f(x)$, to the logistic growth model with differential equation

$$\frac{dy}{dx} = \frac{y(300 - y)}{500},$$

given that $0 < y < 300$ and when $x = 0$, $y = 100$.

4 A scientist monitored the length, L cm, of a particular animal from birth, when it was 51 centimetres long, to being fully grown 25 years later. The scientist found that the length of the animal could be well modelled by a logistic growth model with the differential equation:

$$\frac{dL}{dt} = \frac{1}{500}L(200 - L)$$

where t is the time in years since birth.

a For the logistic growth model with differential equation $\frac{dy}{dt} = ay - by^2$ the limiting value of y is $\frac{a}{b}$. Find the limiting value of L and explain its meaning.

b Using separation of variables and partial fractions find $L(t)$.

c According to the model what was the length of the animal on its 10th birthday?

5 A particular species of lizard is only found to exist naturally on a particular island nature reserve.

Because of favourable breeding conditions and the elimination of feral predators, researchers believe that the current ($t = 0$) population of 160 of these lizards is likely to increase such that P, the population t years from now will follow a logistic growth model with differential equation:

$$\frac{dP}{dt} = \frac{1}{5}P - \frac{1}{12\,500}P^2$$

a Use the technique of separating the variables and partial fractions to determine an expression for P in terms of t for this logistic growth model, giving your answer in the form

$$P = \frac{K}{1 + Ce^{-at}}.$$

b According to this model what is the long-term population limit for this species on this island?

c Determine an estimate for the number of lizards of this species on the island in 10 years' time.

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6 N, the number of fungal units present in a laboratory tray at time t hours, rises from its initial ($t = 0$) value of 200, when counting commenced, such that the rate of change of N is given by

$$\frac{dN}{dt} \approx 0.8N\left(1 - \frac{N}{20\,000}\right).$$

Approximately how many fungal units would be in the tray 8 hours after counting commenced?

Can we get some idea of what the relationship between *x* and *y* will look like from the differential equation directly, without integrating?

Whilst there are a number of methods for solving differential equations, other than separating the variables, not all differential equations are solvable algebraically.

In such cases we might be able to use the differential equation to learn what the graph of the relationship looks like. To help us understand this approach we will initially consider it for differential equations that we *could* algebraically integrate.

Consider the differential equation $\frac{dy}{dx} = 2$.

By integrating we know this has the general solution $y = 2x + c$.

Could we make this general solution apparent without integrating?

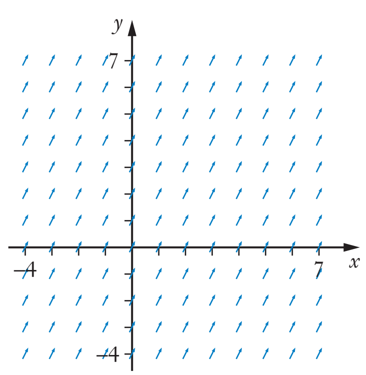
One way is to draw the **slope field** of the differential equation, also called the **direction field** or **gradient field**.

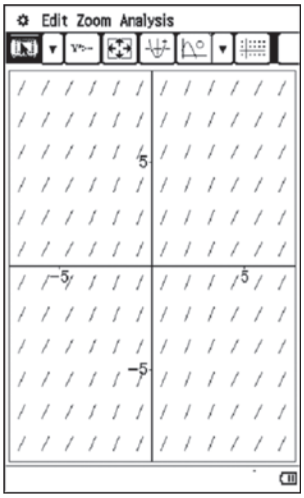
Slope fields

A **slope field** of a differential equation shows the derivative at a given point as a line segment drawn at that point, with the gradient of the line segment equal to the derivative at that point.

For the differential equation $\frac{dy}{dx} = 2$ the gradient is always equal to 2 so we would expect the slope field to show line segments all having a gradient of 2.

This is indeed the case in the diagrams shown, which show the slope field for this differential equation, as a graph below and on a calculator display on the right.





Can you see how the lines of the slope field suggest the family of curves that form the general solution, i.e. $y = 2x + c$?

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Now consider the differential equation $\frac{dy}{dx} = 2(x - 1)$.

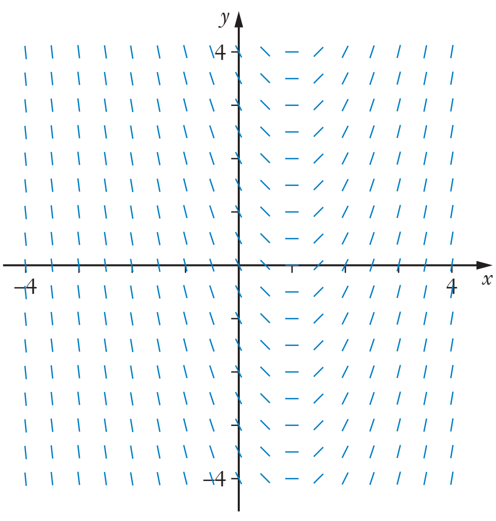
What would we expect the slope field to look like?

Well:
- If $x = 1$ we would expect the gradient to be zero.
- If $x > 1$ we would expect the gradient to be positive and becoming 'larger positive' as x becomes 'larger positive'.
- If $x < 1$ we would expect the gradient to be negative and becoming 'larger negative' as x becomes 'larger negative'.

These expected features are indeed evident on the slope field diagram below.

Note

In the slope field diagram on the right, the gradient lines are 'centred' on their x-value. Some applications will draw them with their 'start' on the x-value. In that case all of the lines shown here would move a little. This need not cause a problem, just be aware that some slope fields may be presented differently.



Again can you see how the lines of the slope field suggest the family of curves that form the general solution, i.e. $y = (x - 1)^2 + c$?

On a calculator capable of displaying slope fields, view the slope field of the differential equation $y' = 2(x - 1)$.

Are there any programs on internet sites capable of displaying slope fields? Investigate.

Exercise 10D

For each of the following differential equations think what the slope field will look like and use your thoughts to make a rough sketch of what you think it will be like.

Then turn the page where all eight slope fields are drawn, plus two extras, and match up each equation with a slope field shown.

1 $\frac{dy}{dx} = 1$

2 $\frac{dy}{dx} + 2 = 0$

3 $\frac{dy}{dx} = 4 - 2x$

4 $\frac{dy}{dx} = x(x - 3)$

5 $\frac{dy}{dx} = (x + 1)(3 - x)$

6 $\frac{dy}{dx} = \sqrt{x}$

7 $\frac{dy}{dx} = 2^x$

8 $\frac{dy}{dx} = \frac{x}{2}$

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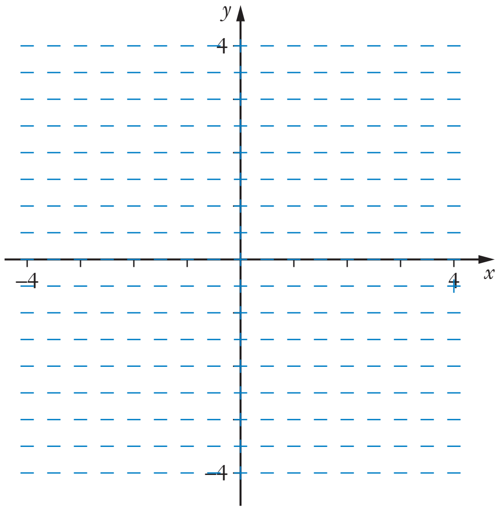

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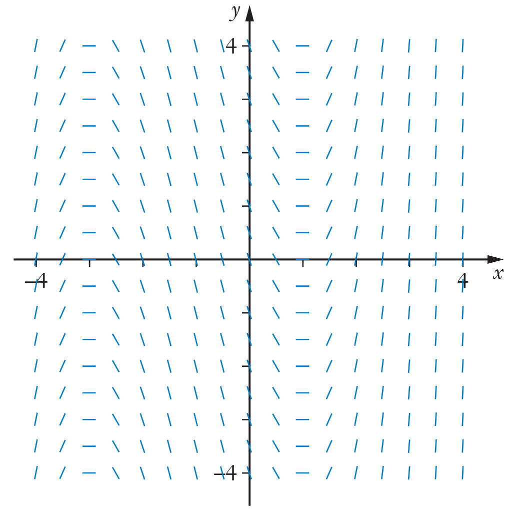

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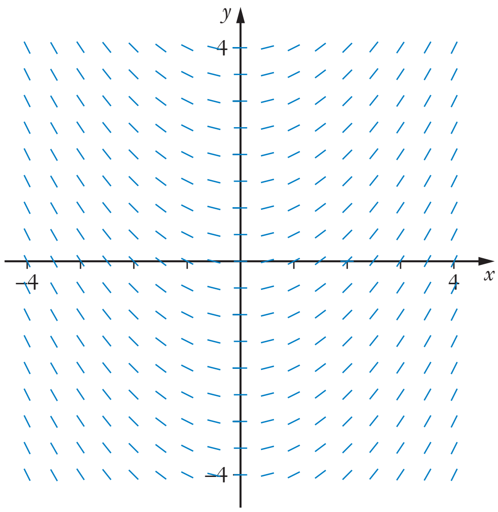

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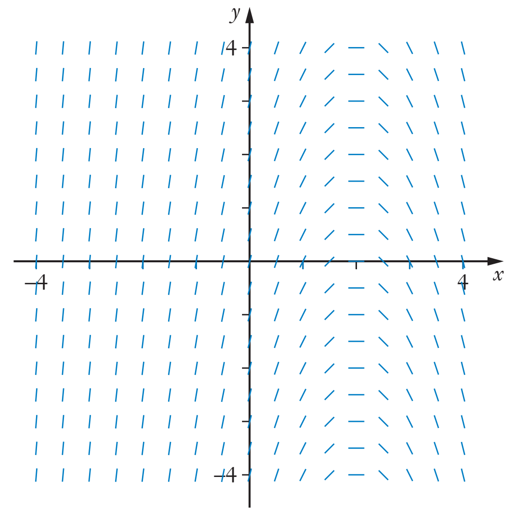

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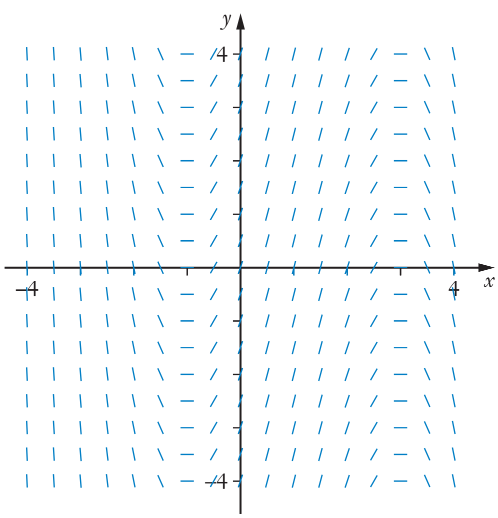

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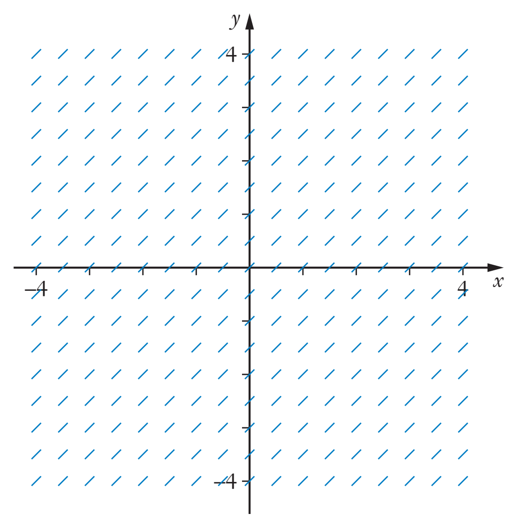

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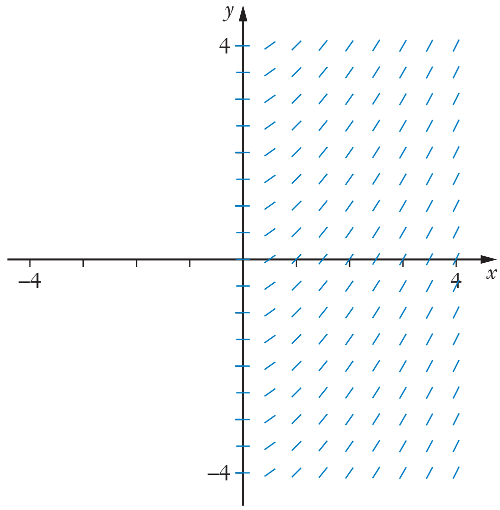

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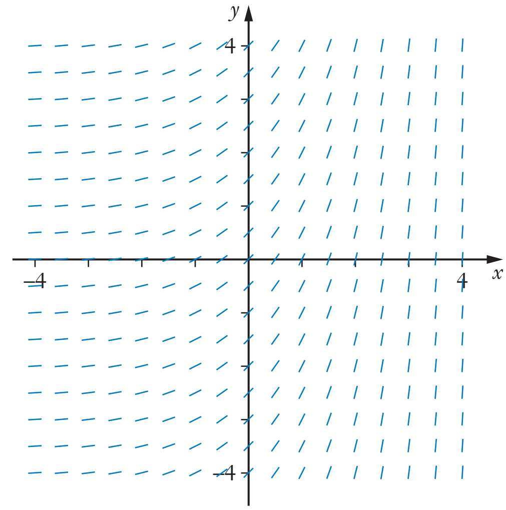

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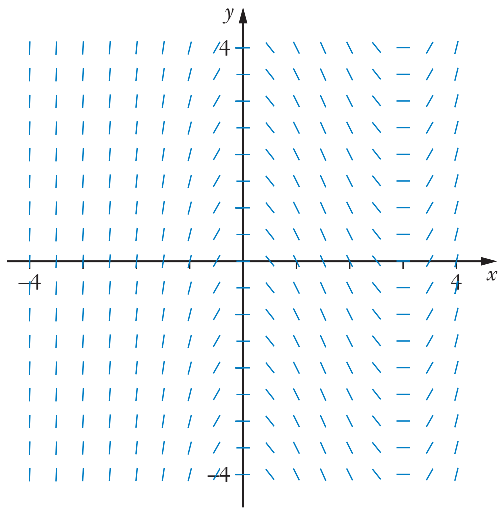

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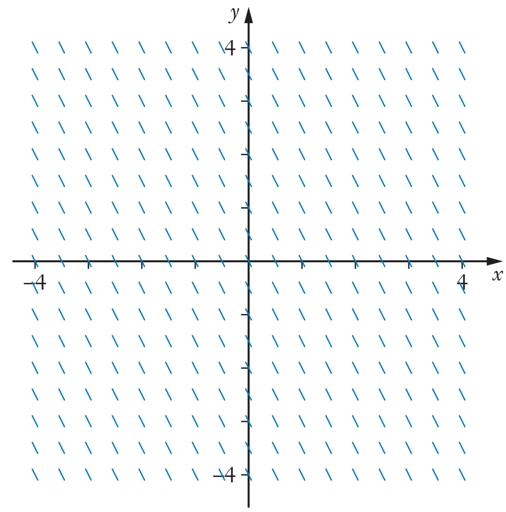


9 Consider the differential equation

$$y' = xy.$$

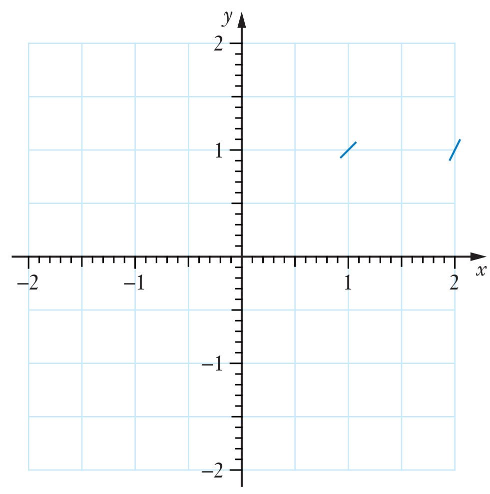
At the point (1, 1) $\quad y' = 1 \times 1$
$\quad\quad = 1$

A small line, with gradient 1, has been drawn on the graph at the point (1, 1).

Similarly a line has been drawn at the point (2, 1) and with gradient 2.

On graph paper draw the axes as shown, i.e. $-2 \le x \le 2$ and $-2 \le y \le 2$, and put small gradient lines at all of the points (a, b) for integer a and b.

View the slope field for $y' = xy$ on a graphic calculator or internet facility.



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Euler's method

We will consider here a numerical method for finding an approximate graphical solution to a differential equation, given some initial conditions. A numerical method would be of most use when solving differential equations that are not easily solved agebraically, and whose slope fields may not be easy to draw. However we will apply the method to a differential equation we could solve algebraically, so that we can see how accurate our solution obtained numerically is.

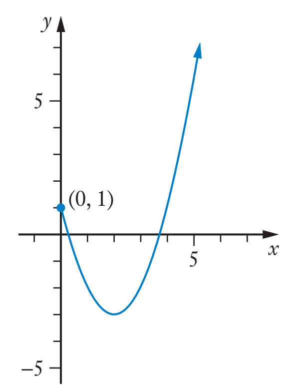
First recall the **small changes formula**, or **incremental formula**, first encountered in *Mathematics Methods* Unit Three, and revised earlier in this book.

δy, the small change in y, caused by δx, the small change in x, can be approximately determined using

$$\delta y \approx \frac{dy}{dx}\delta x.$$

Consider the differential equation $\frac{dy}{dx} = 2x - 4$ for $x \geq 0$, and suppose that when $x = 0$, $y = 1$.

We know that the solution is $\quad y = x^2 - 4x + 1$
which is shown graphed on the right, for $x \geq 0$.



Let us now try to obtain this graph by applying a numerical method to the differential equation.

Starting at our known point (0, 1) we will increase the x-coordinate by δx. Then, using the incremental formula to determine δy, the approximate change in y, we will have a new point $(x + \delta x, y + \delta y)$. We then repeat the process with this new point to give the next point, and so on. Plotting the points will give us an approximate graphical solution to the differential equation.

Let us choose $\delta x = 1$, and so

$$\begin{aligned}\delta y &\approx \frac{dy}{dx}(1)\\ &= (2x - 4)(1)\end{aligned}$$

Now create a table starting at (0, 1) and increasing by $\delta x = 1$ each step.

					(Next point)	
	x	y	δx	$\delta y = (2x - 4)(1)$	$x + \delta x$	$y + \delta y$
From 1st point	0	1	1	$2(0) - 4 = -4$	1	−3
From 2nd point	1	−3	1	$2(1) - 4 = -2$	2	−5
From 3rd point	2	−5	1	$2(2) - 4 = 0$	3	−5
From 4th point	3	−5	1	2	4	−3
From 5th point	4	−3	1	4	5	1
From 6th point	5	1	1	6	6	7

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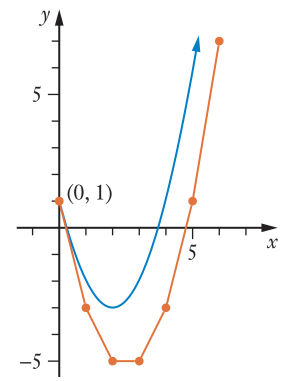
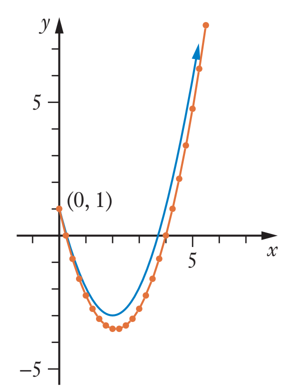
The approximate graphical solution is shown on the right.



Not a very good approximation to what we know the accurate answer should be, you may say. Well, you would be correct but then we were using $\delta x = 1$. If we make our incremental increase in x smaller, our incremental formula will give a better approximation.

The process is repeated below for $\delta x = 0.25$ and, as can be seen, the approximation is now much improved.

x	y	δx	$\delta y = (2x - 4)(0.25)$	$x + \delta x$	$y + \delta y$
0	1	0.25	−1	0.25	0
0.25	0	0.25	−0.875	0.5	−0.875
0.5	−0.875	0.25	−0.75	0.75	−1.625
0.75	−1.625	0.25	−0.625	1	−2.25
1	−2.25	0.25	−0.5	1.25	−2.75
1.25	−2.75	0.25	−0.375	1.5	−3.125
1.5	−3.125	0.25	−0.25	1.75	−3.375
1.75	−3.375	0.25	−0.125	2	−3.5
2	−3.5	0.25	0	2.25	−3.5
2.25	−3.5	0.25	0.125	2.5	−3.375
2.5	−3.375	0.25	0.25	2.75	−3.125
2.75	−3.125	0.25	0.375	3	−2.75
3	−2.75	0.25	0.5	3.25	−2.25
3.25	−2.25	0.25	0.625	3.5	−1.625
3.5	−1.625	0.25	0.75	3.75	−0.875
3.75	−0.875	0.25	0.875	4	0
4	0	0.25	1	4.25	1
4.25	1	0.25	1.125	4.5	2.125
4.5	2.125	0.25	1.25	4.75	3.375
4.75	3.375	0.25	1.375	5	4.75
5	4.75	0.25	1.5	5.25	6.25
5.25	6.25	0.25	1.625	5.5	7.875
5.5	7.875	0.25	1.75	5.75	9.625
5.75	9.625	0.25	1.875	6	11.5



The numerical method used above is known as **Euler's method**.

A computer speadsheet can be very useful when constructing tables like the one above. Some interactive websites display the process.

Produce graphs and tables like those just shown but now for the differential equation

$$\frac{dy}{dx} = 6 - 2x \text{, with } y = -5 \text{ when } x = 0.$$

Consider $0 \le x \le 6$ and use $\delta x = 1$ and then $\delta x = 0.25$.

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Miscellaneous exercise ten

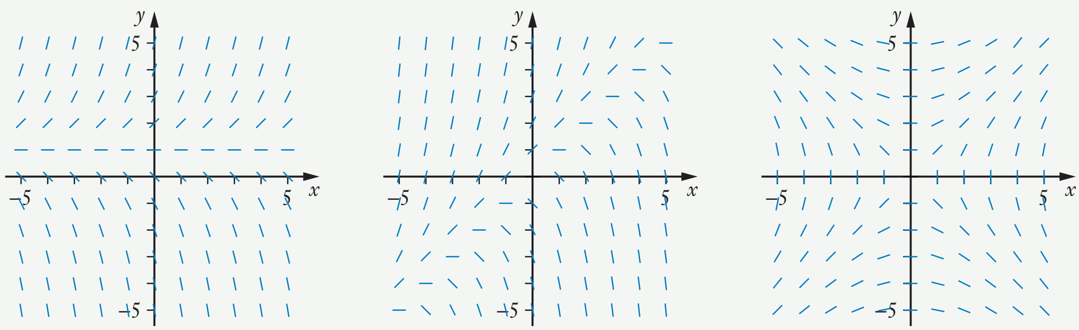
This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters in this unit, and the ideas mentioned in the Preliminary work section at the beginning of the unit.

1 The three slope fields shown below are for the differential equations

$$\frac{dy}{dx} = y - x \qquad \frac{dy}{dx} = \frac{x}{y} \qquad \frac{dy}{dx} = y - 1$$

By considering locations where $\frac{dy}{dx} = 0$ match each equation to a slope field.

Slope field A Slope field B Slope field C



2 In enzymology (the study of enzymes), an important differential equation is the Michaelis-Menten equation, which takes the form:

$$\frac{ds}{dt} = -\frac{Vs}{K+s},$$ where s and t are variables ($s > 0$) and V and K are constants.

Use the method of separation of variables to obtain the general solution

$$s + K \ln s = -Vt + c, \text{ where } c \text{ is some constant.}$$

3 If $\frac{dy}{dx} = e^{x^2}$ use the incremental formula $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$ to find the approximate change in y when x changes from 2.001 to 2.002.

Compare your answer with a calculator evaluation of $\int_{2.001}^{2.002} e^{x^2}\, dx$.

4 Find an expression for $\frac{dy}{dx}$ for each of the following:

a $y = 2 \sin x$

b $y = \sin^2 x$

c $y = \sin(\sin x)$

d $y = \frac{2x+3}{5-3x}$

e $y = (2x+3)^3$

f $2xy + y^3 - 15 = 3 \sin x$

g $x^2 + 3y^2 = y \ln x$

h $5x + 3 \ln(2y+1) = 3xy$

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5 Solve $2y\dfrac{dy}{dx} = e^{2x}$, given that when $x = 0$, $y = 3$.

6 Find the equations of the tangents to the curve

$$y^2 + 5xy + x^2 = 15$$

at the points on the curve where $x = 1$.

7 The area lying in the first quadrant and bounded by the y-axis, the straight lines $y = 2$ and $y = 4$ and the curve $y = x^2$ is rotated one revolution about the y-axis.

Find the volume of the solid so formed.

8 Using the suggested substitution, or otherwise, determine each of the following indefinite integrals algebraically.

a $\displaystyle\int x(3x^2 - 5)^7\,dx \qquad u = 3x^2 - 5$

b $\displaystyle\int x(x - 5)^7\,dx \qquad u = x - 5$

c $\displaystyle\int \frac{8x}{\sqrt{x^2 - 3}}\,dx \qquad u = x^2 - 3$

d $\displaystyle\int 10x\sqrt{5x - 2}\,dx \qquad u = 5x - 2$

e $\displaystyle\int 8x\sin(x^2 - 5)\,dx \qquad u = x^2 - 5$

f $\displaystyle\int e^x(1 + e^x)^4\,dx \qquad u = 1 + e^x$

g $\displaystyle\int \frac{4x}{\sqrt{x - 3}}\,dx \qquad u = x - 3$

h $\displaystyle\int \frac{2x + 1}{(x + 2)^3}\,dx \qquad u = x + 2$

9 Find a formula for the rate of increase in the volume of a sphere when the radius, r cm, is increasing at a constant rate of 0.25 cm/s.

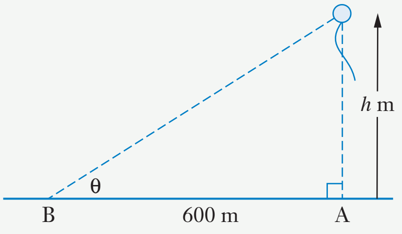
a What is the rate of change of volume when $r = 10$?

b What is the radius of the sphere when the rate of change of volume is 256π cm^3/s?

10 A balloon is released from a point A on horizontal ground and rises vertically.

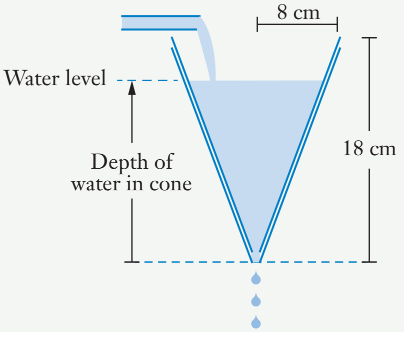
From a point B, 600 metres from A and on the same horizontal level as A, the angle of elevation of the balloon, θ radians, is monitored.

If the balloon rises at a steady 10 m/s find the rate of change of θ, in rad/s, when the balloon is 800 metres above the ground.



11 The diagram shows a funnel in the shape of an upturned cone of height 18 cm and 'base' radius 8 cm.

If water is flowing in at 16 cm^3/s and out at 4 cm^3/s, how fast is the water level rising at the instant that the water in the cone has a depth of 6 cm?



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12 Show that if $x = a\sin u$ then

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = u + c,$$

where c is some constant.

13 Suppose that the function $A(t)$ gives the total worldwide reserve of a particular natural resource at time t years. If this resource is being used at an instantaneous rate of R tonnes/year, A will decrease and so $R = -\frac{dA}{dt}$.

If R itself is continuously increasing at 8% per year then $\frac{dR}{dt} = 0.08R$,

i.e. $R = R_0e^{0.08t}$ where R_0 is the rate of use, in tonnes/year, when $t = 0$.

Thus the rate of change of world reserves of this commodity is given by:

$$\frac{dA}{dt} = -R_0e^{0.08t}$$

If our current instantaneous rate of use is 5 000 000 tonnes per year, find

a the amount we will use during the next ten years,

b in how many years the resource will be exhausted if current world reserves are 200 000 000 tonnes and no new reserves are discovered.

14 One of the worked examples in an earlier chapter determined that the antiderivative of $\cos^4 x$ is

$$\frac{3}{8}x + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c.$$

However, as the display above right suggests, the answer could be

$$\frac{\sin x\cos^3 x}{4} + \frac{3\sin x\cos x}{8} + \frac{3x}{8}.$$

$$\int \left((\cos(x))^4\right) dx$$

$$\frac{\sin(x)\cdot(\cos(x))^3}{4} + \frac{3\cdot\sin(x)\cdot\cos(x)}{8} + \frac{3\cdot x}{8}$$

Show that, with the exception of the fact that the calculator display does not show the constant, these two expressions are the same.

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